Wide-sense Nonblocking Under New Compound Routing Strategies

J.Y. Guo*[1], J.W. Syu[2]

Department of Mathematics, National Taiwan Normal University
11677 No.88 Sec 4, Ting-Chou Rd. Taipei, Taiwan

*[1]davidguo@ ntnu.edu.tw; [2]etra1340@hotmail.com

Abstract

Previous work by Chang et al. showed that the symmetric 3-stage Clos network is SNB if and only if it is WSNB under any of the five strategies: save the unused (STU), packing (P), minimum index (MI), cyclic dynamic (CD), and cyclic static (CS). The work also proved that the multi-$\log_d N$ network is SNB if and only if it is WSNB under the STU, P, CD, and CS strategies. In this paper, the P and STU strategies are extended to six new strategies that give the same number of middle crossbar as SNB. Some special properties of the $\log_d N$ network are also described.

Keywords

WSNB; Nonblocking Network; Routing Strategy

Introduction

The 3-stage network first proposed by Closis now commonly known as the *3-stage Clos network.* It isdenoted as $C(n_1, r_1, m, n_2, r_2)$ and as $C(n, m, r)$ for the symmetric case (See Fig. 1).

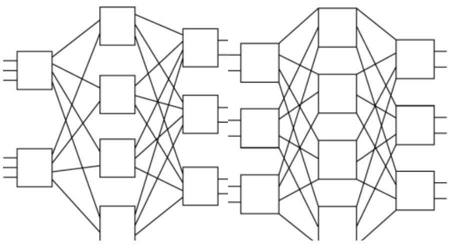


FIG. 1 $C(3, 2, 4, 2, 3)$ and $C(2, 4, 3)$

The $\log_d N$network is a network with n stages, where every stage has d^{n-1} switches withsize $d \times d$ such that the total number of inputs (left side) or outputs (right side) is $d^n = N$. In this network, there is exactly one path from an arbitrary input to an arbitrary output. There are several types of $\log_d N$ networks, such as Omega, Banyan, and Baseline, but they are all equivalent under a permutation of switches in a common stage. The multi-$\log_d N$ network proposed by Lea is a three-stage network, where the first stage consists of N switches with thesize $1 \times p$, the second stage of $p \log_d N$ networks, and the third stage consists of N switches with thesize $p \times 1$ (See Fig. 2). For convenience, the graph of a $\log_d N$ network is converted into the digraph of the Baseline network (See Fig. 3). A three-stage network is strictly nonblocking (SNB) if there are enough middle switches such that a request can always be routed in any state. This is considered wide-sense nonblocking (WSNB) under a routing strategy A if every request is routed under A. Let $p(V)$ be the number of copies such that a network V is SNB if and only if $p \geq p(V)$. Then $p(V) - 1$ is called the maximal blocking number (MBN). Let $M(V, \gamma)$ be the set of such $p(V) - 1$ requests blocking γ. Note that the requests in $M(V, \gamma)$ are not unique.

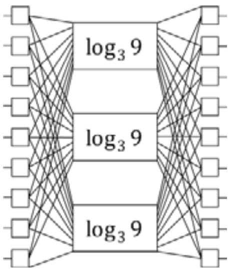


FIG. 2 Multi-$\log_3 9$ NETWORK

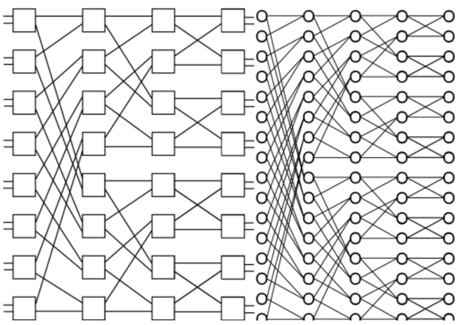


FIG. 3 The $\log_2 16$ network and it digraph

The five frequently applied strategies are:

1. Packing (P): Route through the busiest middle switches.
2. Save-the-unused (STU): Do not route through currently unused switches unless there is no other choice.
3. Cyclic dynamic (CD): If the lastest request was routed through switch M_j, try $M_{j+1}, M_{j+2}, \ldots$ in

cyclic order.

4. Cyclic static (CS): If the lastest request was routed through switch M_j, try $M_j, M_{j+1}, \dots$ in cyclic order.
5. Minimum index (MI): Route through switch M_j where the index j is as small as possible.

Note that under CD, CS and MI strategies each selection of middle switches is unique, while under P and STU there can be many feasible possibilities to select a middle switch to route. Consequently, arguments in proving lower bounds for the number of required middle switches under P or STU usually assume that the most favorable middle switch has been choosen if there are more than one choice and then leads to a blocking status. However, it is still unknown whether the network is blocked or not when a smart selection-strategy is designated to the multiple-choice situation. In this paper, STU and P are strengthened to six new strategies for the 3-stage Clos network and it is proved that the strengthened strategies perform as good as the usual P and STU, that is, the known lower bounds for the number of required middle switches under P and STU remain lower bounds for the strengthened strategies.

The Symmetric 3-stage Clos Network $\boldsymbol{C(n,m,r)}$

For the symmetric case, Closproved that the 3-stage Clos network $C(n,m,r)$, where $r \geq 2$, is SNB if and only if $m \geq 2n-1$. Benesfurther proved that the 3-stage Clos network $C(n,m,2)$ is WSNB under STU if $m \geq \lfloor 3n/2 \rfloor$ .Smithshowed that the condition $m \geq \lfloor 3n/2 \rfloor$ should be satisfied by proving that $C(n,m,r)$is not WSNB under P or MI if$m < 2n - \lfloor n/r \rfloor$. Since P is equal to STU as $r=2$, it follows immediately that $C(n,m,2)$ is WSNB under STU or P if and only if $m \geq \lfloor 3n/2 \rfloor$. For the case $r=3$, Du et al.proved that $C(n,m,r)$ is WSNB under P or STU if and only if it is SNB, i.e. $m \geq 2n-1$. Chang et al. simplified this proof using a sequence of 3×3 Paull matrices. In a Paull matrix, the rows and columns represent input and output switches respectively, every cell (i,j) represents a request from input switch I_i to output switch O_j, and the numbers in cell (i,j) denote the middle switches in the routing path. For example, See Fig. 4, two requests are routed from I_1 to O_2 through the second and fourth middle switches. STU and P are strengthened to the following six routing strategies:

1. STU+MI: Routing is done mainly using STU. If there are several choices available under STU, try strategy MI.
2. STU+CD: Routing is done mainly using STU. If there are several choices available under STU, try strategy CD.
3. STU+CS: Routing is done mainly using STU. If there are several choices available under STU, try strategy CS.
4. P+MI: Routing is done mainly using P. If there are several choices available under STU, try strategy MI.
5. P+CD: Routing is done mainly using P. If there are several choices available under STU, try strategy CD.
6. P+CS: Routing is done mainly using P. If there are several choices available under STU, try strategy CS.

For P+CD or STU+CD, the most recent routing middle switch is assumed to be M_j. If we have m switches in total,$M_{i_1}, M_{i_2}, M_{i_3}, \dots, M_{i_m}$etc., to route under P or STU, and if $i_k \leq j < i_{k+1}$, then we try $M_{i_{k+1}}, M_{i_{k+2}}, \dots$ based on the cyclic order.

For P+CS or STU+CS, the strategy is similar to CD except that if $j = i_k$, we try M_{i_k}at first.

By definition, it is easily seen that WSNB under P+MI, P+CD and P+CS implies WSNB under STU+ MI, STU+CD and STU+CS respectively. Therefore, we only need to prove the bound for strategies P+MI, P+CD and P+CS.

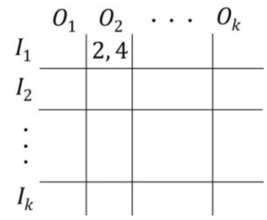
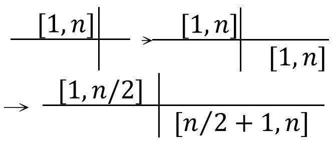


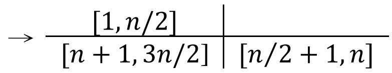
FIG. 4 PAULL MATRIX

Theorem 2.1 *The 3-stage Clos network $C(n,m,2)$, is WSNB under P(STU)+CD, P(STU)+CS, P(STU)+MI if and only if* $m \geq \lfloor 3n/2 \rfloor$.

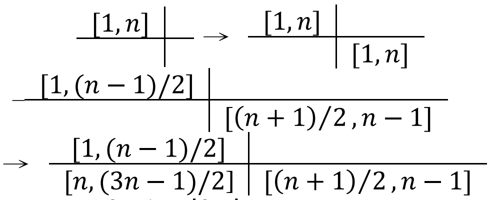
Proof. If $m \geq \lfloor 3n/2 \rfloor$, $C(n,m,2)$ is WSNB under P; this implies that $C(n,m,2)$ is WSNB under these three strategies. To prove the "only if" condition, an example of a sequence of requests has been constructed such that the network is blocked when $m < \lfloor 3n/2 \rfloor$. The sequence of requests is depicted as Paull matrices.

For$n$ even,





For n odd,



Since n is odd, $\frac{3n-1}{2} = \left\lfloor \frac{3}{2} n \right\rfloor$.

For the end of these two cases, $\left\lfloor \frac{3}{2} n \right\rfloor$ middle switches is used under all of these three strategies, then Theorem 2.1 is proved. □

It isnoted here that the proof in Theorem 2.1 is also applicable to the following theorem: $C(n, m, 2)$ is WSNB under P only if $m \geq \left\lfloor \frac{3}{2} n \right\rfloor$.

For the remaining four strategies, P(STU)+CD and P(STU)+CS, we start proving a useful lemma:

Lemma 2.2 *If a state s can be reached under P(STU), then s can be reached under P(STU)+CD or P(STU)+CS.*

Proof. Consider a general state s'. If state s'' can be obtained from s' under P by disconnecting a request, it can also be obtained from s' under P+CD or P+CS. Assume that s'' is obtained from s' under P by adding a request routed through a middle switch M_k. If M_k is the only choice under P, it is also the sole choice under P+CD or P+CS. Supposing that there are several middle switches, i.e. $M_{i_1}, M_{i_2}, \dots, M_{i_l}$, that can be routed through under P, where $k \in \{i_1, i_2, \dots, i_l\}$. The following two cases apply:

Case I: If P+CD assigns a request through M_{i_h}, where $i_h = k$, no further steps are required. Otherwise the request is deleted and generated again. P+CD will assign it through M_{i_h+1}. This step is repeated until P+CD assigns the request through M_k.

Case II: If P+CS assigns two requests through M_{i_h} and M_{i_h+1}, where $i_h = k$ or $i_{h+1} = k$, no further steps are required. Otherwise the requests are disconnected and generated again. Then P+CS assigns them through M_{i_h+1} and M_{i_h+2}. This step is repeated until P+CS assigns one of them through M_k.

Since s' is arbitrary and s'' is obtained under P, so as STU. □

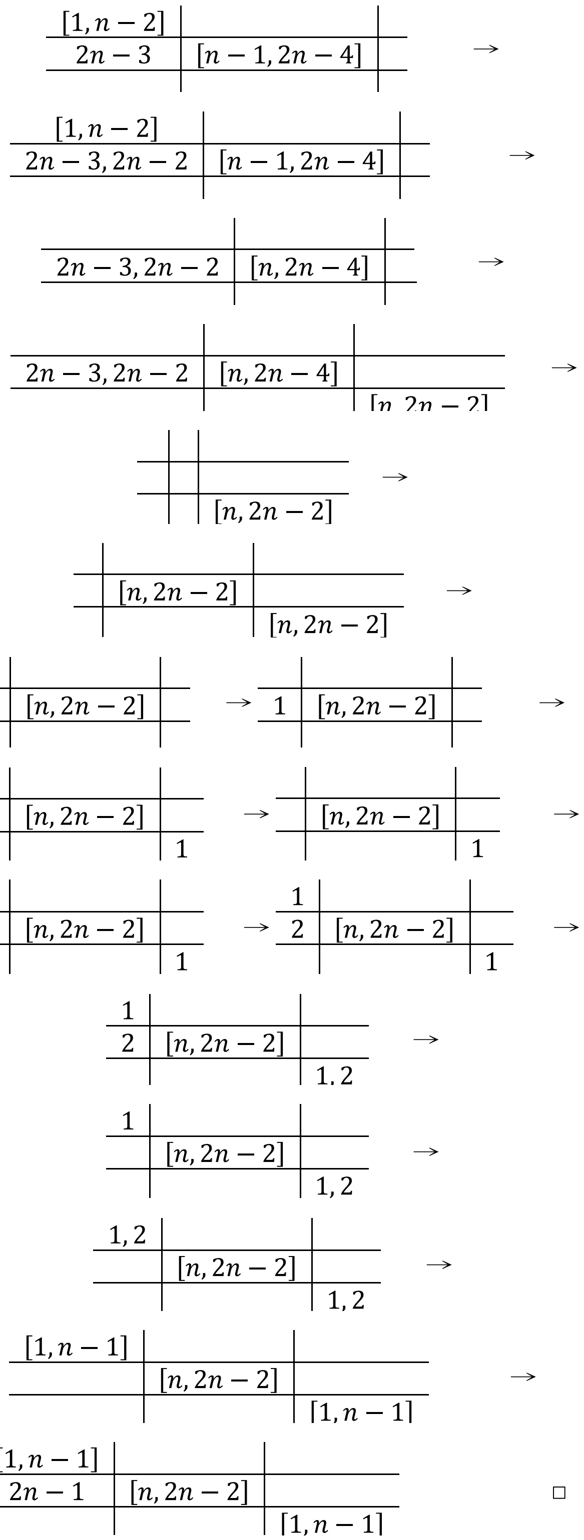
Theorem 2.3 *The 3-stage Clos network $C(n, m, r)$, where $r \geq 3$, is WSNB under any P(STU)+CD and P(STU)+CS if and only if $m \geq 2n-1$.*

Proof. This theorem is immediately followed by Lemma 2.2. □

Theorem 2.4*The 3-stage Clos network $C(n, m, r)$, where $r \geq 3$, is WSNB under P(STU)+MI if and only if $m \geq 2n-1$.*

Proof. Proving the "if" part of the theorem is trivial; we just need to prove the "only if" part.

By induction of$n$, it is assumed that we have



□

The Asymmetric 3-stage Clos Network $\boldsymbol{C(n_1, r_1, m, n_2, r_2)}$

In this section, the asymmetric case is discussed. For the asymmetric case, Chang et al.proved the following two theorems:

Theorem 3.1 *Suppose* $n_1 \geq n_2$. *Then* $C(n_1, 2, m, n_2, 2)$ *is WSNB under P or STU if and only if* $m \geq \min\{2n_2, n_2 + \lfloor n_1/2 \rfloor\}$.

Theorem 3.2 *Suppose* $n_1 \geq n_2$, $\min\{r_1, r_2\} \geq 2$ *and* $\max\{r_1, r_2\} \geq 3$. *Then the 3-stage Clos network* $C(n_1, r_1, m, n_2, r_2)$ *is WSNB under P or STU if and only if* $m \geq \min\{r_2 n_2, n_1 + n_2 - 1\}$.

Similarly, in order to have one and only one choice of middle crossbar at every step, the following strategies are employed: P(STU)+CD, P(STU)+CS, P(STU)+MI.

Theorem 3.3 *Suppose* $n_1 \geq n_2$, $\min\{r_1, r_2\} \geq 2$ *and* $\max\{r_1, r_2\} \geq 3$. *Then the 3-stage Clos network* $C(n_1, r_1, m, n_2, r_2)$ *is WSNB under P(STU)+CD or P(STU)+CS if and only if* $m \geq \min\{r_2 n_2, n_1 + n_2 - 1\}$.

Proof. By Lemma 2.2 and Theorem 3.2, this theorem is proved. □

Note that the bound $r_2 n_2$ is a trivial bound, and therefore, we only need to prove the bound $n_1 + n_2 - 1$.

Lemma 3.4 *Suppose* $n_1 \geq n_2$. *Then the 3-stage Clos network* $C(n_1, 2, m, n_2, 3)$ *is WSNB under P(STU)+MI if and only if* $m \geq \min\{r_2 n_2, n_1 + n_2 - 1\}$.

Proof. Let $A = [1, n_1 - n_2 - 1]$and $B = [n_1 - n_2, n_1 - 2]$, and follow the following steps:

$$\begin{array}{c|c|c} A & B & n_1-1 \\ \hline & & \end{array} \to \begin{array}{c|c|c} & & n_1-1 \\ \hline & & \end{array} \to$$

$$\begin{array}{c|c|c} & & \\ \hline n_1-1 & & \end{array} \to \begin{array}{c|c|c} & n_1-1 & \\ \hline & & \end{array} \to$$

$$\begin{array}{c|c|c} & & \\ \hline & & n_1-1 \end{array} \to \begin{array}{c|c|c} & & 1 \\ \hline & & n_1-1 \end{array} \to$$

$$\begin{array}{c|c|c} & & \\ \hline & 1 & n_1-1 \end{array} \to \begin{array}{c|c|c} 1 & & \\ \hline & & n_1-1 \end{array} \to$$

$$\begin{array}{c|c|c} 1 & & 2 \\ \hline & & n_1-1 \end{array} \to \ldots \to \begin{array}{c|c|c} A & B & \\ \hline & & n_1-1 \end{array} \to$$

$$\begin{array}{c|c|c} A & B & n_1 \\ \hline & & n_1-1 \end{array} \to \begin{array}{c|c|c} & & n_1 \\ \hline & & n_1-1 \end{array} \to$$

$$\begin{array}{c|c|c} & & \\ \hline & n_1 & n_1-1 \end{array} \to \begin{array}{c|c|c} n_1 & & \\ \hline & & n_1-1 \end{array} \to$$

$$\begin{array}{c|c|c} & & \\ \hline & & n_1-1, n_1 \end{array} \to \ldots \to \begin{array}{c|c|c} A & B & \\ \hline & & n_1-1, n_1 \end{array}$$

$$\to \ldots \to \begin{array}{c|c|c} A & B & \\ \hline & & [n_1-1, n_1+n_2-3] \end{array} \to$$

$$\begin{array}{c|c|c} A & B & n_1+n_2-2 \\ \hline & & [n_1-1, n_1+n_2-3] \end{array} \to$$

$$\begin{array}{c|c|c} & & n_1+n_2-2 \\ \hline & & [n_1, n_1+n_2-3] \end{array} \to \ldots \to$$

$$\begin{array}{c|c|c} A & B, n_1-1 & \\ \hline & & [n_1, n_1+n_2-2] \end{array} \to$$

$$\begin{array}{c|c|c} A & B, n_1-1 & n_1+n_2-1 \\ \hline & & [n_1, n_1+n_2-2] \end{array}$$

□

Lemma 3.5*Suppose* $n_1 \geq n_2$. *Then the 3-stage Clos network* $C(n_1, 3, m, n_2, 2)$ *is WSNB under P(STU)+MI if and only if* $m \geq \min\{r_2 n_2, n_1 + n_2 - 1\}$.

Proof. Applying a similar argument as that in Lemma 3.4, we can reach the state

$$\begin{array}{c|c} [1, n_2-1] & \\ \hline n_1+n_2-1 & [n_2, n_1+n_2-2] \end{array}$$

□

Theorem 3.6 *Suppose* $n_1 \geq n_2$, $\min\{r_1, r_2\} \geq 2$ *and* $\max\{r_1, r_2\} \geq 3$. *Then the 3-stage Clos network* $C(n_1, r_1, m, n_2, r_2)$ *is WSNB under P(STU)+MI if and only if* $m \geq \min\{r_2 n_2, n_1 + n_2 - 1\}$.

Proof. We need only to prove the two cases "$r_1 = 2, r_2 \geq 3$" and "$r_1 \geq 3, r_2 = 2$". For the former case, "$r_1 = 2, r_2 \geq 3$", let $A = [n_1 - 2n_2 + 1, n_1 - n_2 - 1]$and $B = [n_1 - n_2, n_1 - 2]$. Starting from the state

$$\begin{array}{c|c|c|c|c|c} [1, n_2] & [n_2+1, 2n_2] & \ldots & A & B & \\ \hline & & & & & \end{array} .$$

It is sufficient to route the rest of the requests in the last three columns. By Lemma 3.4, we can reach the state

$$\begin{array}{c|c|c|c|c|c} [1, n_2] & [n_2+1, 2n_2] & \ldots & A & B, n_1-1 & n_1+n_2-1 \\ \hline & & & & & [n_1, n_1+n_2-2] \end{array}$$

under P+MI.

Similarly, for the case "$r_1 \geq 3, r_2 = 2$", we can reach the state

$$\begin{array}{c|c} [1, n_2-1] & \\ \hline n_1+n_2-1 & [n_2, n_1+n_2-2] \end{array}$$

under P+MI by Lemma 3.5. □

The Multi-$\boldsymbol{\log_d N}$ Network

In this section, our attention is turned to the multi-

$\log_d N$ network. Combining the sufficiency proved by Hwang and the necessity proved by Chang et al., we obtain:

Theorem 4.1*The multi-*$\log_d N$ *network* L *is SNB if and only if* $p \geq p(L)$, *where*

$$p(L) = \begin{cases} 2 \times d^{\frac{n-1}{2}} - 1 & \text{for } n \text{ odd,} \\ (d+1) \times d^{\frac{n}{2}-1} - 1 & \text{for } n \text{ even.} \end{cases}$$

It is also knownthat the multi-$\log_d N$ network is WSNB under P or STU if and only if it is SNB. Together with Lemma 2.2, we have the following result.

Theorem 4.2. *The multi-*$\log_d N$ *network is WSNB under P(STU)+CD or P(STU)+CS if and only if it is SNB.*

Proof. Consider the copies of the multi-$\log_d N$ network as the middle switches in the 3-stage Clos networks. This theorem can be proved similar to Lemma 2.2. □

To present our results for P(STU)+MI, a series of properties of the multi-$\log_d N$ network is required.

Theorem 4.3 *The* $\log_d N$ *network can be divided into* $d^{\frac{n}{2}-1}$ *disjoint induced subnetworks* $S_1, S_2, \dots, S_{d^{\frac{n}{2}-1}}$, *where all of them are isomorphic.*

Proof. Since all the $\log_d N$ networks are equivalent, we discuss only the Baseline network. Let $I_k = \left\{(k-1)d^{\frac{n}{2}+1}, (k-1)d^{\frac{n}{2}+1}+1, \cdots, kd^{\frac{n}{2}+1}-1\right\}$ be the subsets of inputs and $O_k = \left\{(k-1)d^{\frac{n}{2}+1}, (k-1)d^{\frac{n}{2}+1}+1, \cdots, kd^{\frac{n}{2}+1}-1\right\}$ the subsets of outputs, where $k = 1, 2, \dots, d^{\frac{n}{2}-1}$, and furthermore, let $S_k = \{I_k, O_k\}$ be the set of all connections from I_k to O_k. By the connection property of the Baseline network, the induced subnetworks$S_1, S_2, \dots, S_{d^{\frac{n}{2}-1}}$ disjoint and have the same configuration. □

By relabeling all the inputs and outputs in S_k as $\left\{0, 1, \dots, d^{\frac{n}{2}+1}-1\right\}$, input 0 of $S_k$is the input $(k-1)d^{\frac{n}{2}+1}$ of L and so on. Let γ_i be a request in S_i and $M(S_i, \gamma_i) = u_{i1}, u_{i2}, \dots, u_{i(p(S_i)-1)}$ be the maximal blocking set. Let the connection $(p,q)_k$ represent the relabeled connection (p,q)in S_k.

Lemma 4.4. *Under the strategies P(STU)+MI, if there are* l *requests through* S_1 *intersecting at a certain node, we can route* $u_{21}, u_{22}, \dots, u_{2l} \in M(S_2, \gamma_2)$ *in the* $1^{st}, 2^{nd}, \dots, l^{th}$ *copies, respectively, and then route* γ_2 *in the* $(l+1)^{th}$ *copy.*

Proof. Let $u_{11}, u_{12}, \dots, u_{1l}$ be the l requests through S_1 which intersect at a particular node. Disconnect all the requests in S_1 except the request in the l^{th} copy and then route u_{2l} in S_2. Route $u_{11}, u_{12}, \dots, u_{1l}$ and then disconnect all the requests in S_1 except the request in the $(l-1)^{th}$ copy such that $u_{2(l-1)}$ can be routed in the $(l-1)^{th}$ copy. By this process, we can route $u_{21}, u_{22}, \dots, u_{2l}$ in the $1^{st}, 2^{nd}, \dots, l^{th}$ copies, respectively, and then route γ_2 in the $(l+1)^{th}$ copy. □

Lemma 4.5. *Under the strategies P(STU)+MI if we can route the* $l+i-1$ *requests* $u_{i1}, u_{i2}, \dots, u_{i(l+i-2)}, \gamma_i$ *in the* $1^{st}, 2^{nd}, \dots, (l+i-1)^{th}$ *copies, respectively, then we can route* $u_{(i+1)1}, u_{(i+1)2}, \dots, u_{(i+1)(l+i-1)}, \gamma_{i+1}$ *in the* $1^{st}, 2^{nd}, \dots, (l+i)^{th}$ *copies, respectively.*

Proof. This lemma is proved by induction on i. For $i = 2$, by Lemma 13, we can route the $l+1$ requests $u_{21}, u_{22}, \dots, u_{2l}, \gamma_2$ in the $1^{st}, 2^{nd}, \dots, (l+1)^{th}$ copies, respectively. First, we route u_{31} in the 1^{st} copy and then disconnect u_{21}. Next, we route $u_{11}, u_{12}, \dots, u_{1l}$ and then delete all the requests in S_1 except the request in the 2^{nd} copy. Third, we route u_{32} in the 2^{nd} copy and then disconnect u_{22}. Through this process, we can easily route $u_{31}, u_{32}, \dots, u_{3l}$ in the $1^{st}, 2^{nd}, \dots, l^{th}$ copies, respectively, and then route γ_3 in the $(l+1)^{th}$ copy. Note that there are two requests γ_2, γ_3 in the $(l+1)^{th}$ copy such that the $(l+1)^{th}$ copy is the busiest, and therefore, we can route γ_1 in the $(l+1)^{th}$ copy. Finally, we delete γ_3 and route $u_{3(l+1)}$ in the $(l+1)^{th}$ copy such that γ_3 must be routed in the $(l+2)^{th}$ copy.

Assume that the statement is true for $i = k$, i.e. we can route the $l+k-1$ requests $u_{k1}, u_{k2}, \dots, u_{k(l+k-2)}, \gamma_k$ in the $1^{st}, 2^{nd}, \dots, (l+k-1)^{th}$ copies. First, we route $u_{(k+1)1}$ in the 1^{st} copy and then disconnect u_{k1}. Second, we route $u_{(k-1)1}, u_{(k-1)2}, \dots, u_{(k-1)(l+k-3)}, \gamma_{k-1}$ in the $1^{st}, 2^{nd}, \dots, (l+k-2)^{th}$ copies and then delete all the requests in S_{k-1} except the request in the 2^{nd} copy. Third, we route $u_{(k+1)2}$ in the 2^{nd} copy and then disconnect u_{k2}. Following this process, we can route the requests $u_{(k+1)1}, u_{(k+1)2}, \dots, u_{(k+1)(l+k-2)}$ in the $1^{st}, 2^{nd}, \dots, (l+k-2)^{th}$ copies, respectively, and then route γ_{k+1} in the $(l+k-1)^{th}$ copy. Note that there are two requests γ_k, γ_{k+1} in the $(l+k-1)^{th}$ copy such that the $(l+k-1)^{th}$ copy is the busiest. Therefore, we can route γ_{k-1} in the $(l+k-1)^{th}$ copy. Finally, we delete γ_{k+1} and route $u_{(k+1)(l+k-1)}$ in the $(l+k-1)^{th}$ copy such that γ_{k+1} must be routed in the $(l+k)^{th}$ copy. □

With Lemmas 4.4 and 4.5, we obtain the following lemma:

Lemma 4.6. *For*$n = \log_d N$*even,themulti-*$\log_d N$ *network is WSNB under P(STU)+MI if and only if it is SNB.*

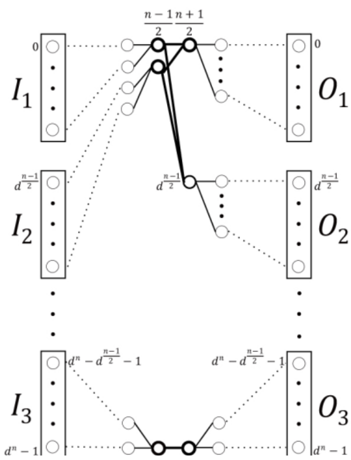
Proof. It suffices to prove the "only if" part of the lemma. By Theorem 4.3, we can partition the multi-

$\log_d N$ network into $d^{\frac{n}{2}-1}$ and then disjoint subnetworks $S_1, S_2, \ldots, S_{d^{\frac{n}{2}-1}}$. By Lemma 4.5, since we can route $(0,0)_1, (1,1)_1, \ldots, \left(d^{\frac{n}{2}}-1, d^{\frac{n}{2}}-1\right)_1, \gamma_2$ in the first $d^{\frac{n}{2}}+1$ copies, respectively, the request $\gamma_{d^{\frac{n}{2}-1}}$ is forced to route through the $\left(d^{\frac{n}{2}}+d^{\frac{n}{2}-1}-1\right)^{th}$ copy, i.e. the $\left((d+1)d^{\frac{n}{2}-1}-1\right)^{th}$ copy. Therefore the condition $p \geq (d+1)d^{\frac{n}{2}-1}-1$ is necessary. □

Next, we will discuss the multi-$\log_d N$ network, where $n = \log_d N$ is odd.

Lemma 4.7. *For* $n = \log_d N$ *odd, the multi-*$\log_d N$ *network is WSNB under P+MI or STU+MI if and only if it is SNB.*

Proof. We select three subsets I_1, I_2 and I_3 of inputs and three subsets O_1, O_2 and O_3 of outputs. Let $I_1 = O_1 = \left\{0, 1, \ldots, d^{\frac{n-1}{2}}-1\right\}$, $I_2 = O_2 = \left\{d^{\frac{n-1}{2}}, d^{\frac{n-1}{2}}+1, \ldots, 2d^{\frac{n-1}{2}}-1\right\}$ and $I_3 = O_3 = \left\{d^n - d^{\frac{n-1}{2}}-1, \ldots, d^n-1\right\}$. In this setting, every request from I_1 to $O_1 \cup O_2$ must intersect at node 0 of the $\left(\frac{n-1}{2}\right)^{th}$ stage, and every request from I_2 to $O_1 \cup O_2$ must intersect at node 1 of the $\left(\frac{n-1}{2}\right)^{th}$ stage. Similarly, every request from $I_1 \cup I_2$ to O_1 must intersect at node 0 of the $\left(\frac{n+1}{2}\right)^{th}$ stage and every request from $I_1 \cup I_2$ to O_2 must intersect at node $d^{\frac{n-1}{2}}$ of the $\left(\frac{n+1}{2}\right)^{th}$ stage. It isnoted that I_1 and I_2 are input switches, O_1 and O_2 are output switches and the bipartite graph comprises nodes 0 and 1 of the $\left(\frac{n-1}{2}\right)^{th}$ stage and nodes 0 and $d^{\frac{n-1}{2}}$ of the $\left(\frac{n+1}{2}\right)^{th}$ stage as a middle switch, and the subnetwork $\{I_3, O_3\}$ is independent to the subnetwork consisting of I_1, I_2, O_1 and O_2 (See Fig. 5).



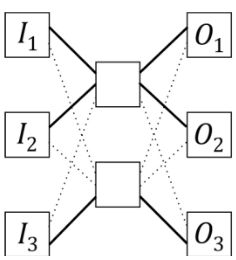


FIG. 5. THE FIGURE ON THE LEFT IS AN INDUCED GRAPH OF THE MULTI-LOG$_d$ N NETWORK, WHERE n IS ODD. THE FIGURE ON THE RIGHT IS THE 3-STAGE CLOS NETWORK $C\left(d^{\frac{n-1}{2}}, 2, 3\right)$ FOR COMPARISON.

We match these to the 3-stage Clos network $C\left(d^{\frac{n-1}{2}}, 2, 3\right)$, and therefore a request (i,j) in $C\left(d^{\frac{n-1}{2}}, 2, 3\right)$ routed through a middle switch M_k corresponds to a request (i,j) in multi-$\log_d N$ routed through the k^{th} copy. By Theorem 2.4, the multi-$\log_d N$ network is WSNB under P(STU)+MI only if $p \geq 2 \times \left(d^{\frac{n-1}{2}}\right) - 1 = 2 \times d^{\frac{n-1}{2}} - 1$. □

By Lemma 4.6 and Lemma 4.7, we have:

Theorem 4.8. *The multi-*$\log_d N$ *network is WSNB under P(STU)+MI if and only if it is SNB.*

Conclusions

In this work with the aim to provide a formal proof in future work, the frequently used strategies P and STU have been strengthened, and thenunfer which wide-sense nonblocking has been discussed. Evidently, the strategies do not make the results better than SNB not only for the 3-stage Clos network but also for the multi- $\log_d N$ network, except for the networks $C(n,m,2)$ and $C(n_1,2,m,n_2,2)$. Therefore, for all 1-1 networks, our understanding suggests that the bound for WSNB is the same as the bound for SNB. .

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